

Perturbative Approach to the Gravitational Lensing by a Non-spherically Distorted Compact Object

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We investigate the gravitational lens effect caused by a non-spherically distorted compact object. The non-spherical property of the gravitational potential is modeled by a quadrupole moment. Under the assumption that the quadrupole contribution is small, we solve perturbatively the lens equation and obtain the image positions and the amplification factors. We show that the separation angle of two major images is only slightly changed by the existence of the quadrupole contribution, whereas the difference of the amplification factors may be significantly modified. Our results indicate that even a tiny non-spherical distortion of the lens potential may cause significant amount of flux anomalies in the lensed images.

§1. Introduction

The gravitational lens is an important tool in astrophysics for probing mass distributions and determining the cosmological parameters.¹⁾ Although previous studies of gravitational lensing have been mostly based on simple, spherically symmetric lens models, some generalizations which include rotation^{2)–6)} and higher-order general relativistic effects^{7),8)} have also been developed. In this paper, we investigate another generalization, i.e., the effect of non-spherical distortion of the gravitational potential in a compact lens object on the gravitational lensing.

A pioneering work on the non-spherically deformed compact lens was done by Asada.⁹⁾ He studied analytically the gravitational lensing caused by a non-spherically deformed star, where the non-spherical property of the gravitational potential was modeled by a quadrupole moment. He employed a rigorous analytic approach and presented solutions of the image positions for a source on the principal axis, an expression of the caustics and the critical curves. However, because of the highly non-linear nature of the lens equation, general solutions of the image positions for a off-axis source and their amplification factors have not been presented.

In this paper, we take a more practical approach, namely the perturbative approximation, and solve analytically the lens equation and obtain the approximate solutions up to the lowest order of the quadrupole moment. We also calculate the Jacobian of the lens mapping and indicate how the quadrupole moment changes the amplification factors.

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§2. Lens equation

In this section, we summarize the lens equation for a compact object with a quadrupole moment. See also Asada⁹⁾ for detail. The gravitational potential of a non-spherically deformed compact object is modeled by use of a monopole and a quadrupole moment:

$$\phi = \phi_0 + \phi_2 = -\frac{GM}{r} - \frac{G}{2} \left(3 \frac{x^i x^j}{r^5} - \frac{\delta^{ij}}{r^3} \right) I_{ij}, \quad (2.1)$$

where

$$M = \int \rho d^3x, \quad I_{ij} = \int \rho x_i x_j d^3x, \quad (2.2)$$

and ρ is the mass density of the lens object. Using the above form of the potential, the deflection angle 2-vector α is calculated as follows:

$$\alpha^i = \frac{4GM}{c^2} \frac{\xi^i}{|\xi|^2} + \frac{8G}{c^2} \left(2Q_{jk} \frac{\xi^j \xi^k \xi^i}{|\xi|^6} - Q_{ij} \frac{\xi^j}{|\xi|^4} \right), \quad (2.3)$$

where the 2-dimensional vector ξ denotes the image position, and

$$Q_{ij} = \int \rho \left(X_i X_j - \frac{1}{2} \delta_{ij} |\mathbf{X}|^2 \right) d^3X \quad (2.4)$$

denotes the trace-free quadrupole moment. Without loss of generality, we can assume the normalized quadrupole moment is diagonalized as follows,

$$\tilde{Q}_{ij} \equiv \frac{c^2 D_S}{2GM^2 D_L D_{LS}} Q_{ij} = \begin{pmatrix} e & 0 \\ 0 & -e \end{pmatrix}, \quad (2.5)$$

where D_S, D_L, D_{LS} are the angular diameter distances from the observer to the source, from the observer to the lens, and from the lens to the source, respectively. Hereafter, we assume $e > 0$. Finally, the lens equation is

$$\beta_x = x - \frac{x}{x^2 + y^2} - e \frac{(x^2 - 3y^2)x}{(x^2 + y^2)^3}, \quad (2.6)$$

$$\beta_y = y - \frac{y}{x^2 + y^2} - e \frac{(3x^2 - y^2)y}{(x^2 + y^2)^3}, \quad (2.7)$$

where $\beta = (\beta_x, \beta_y)$ and $\theta = (x, y)$ are the source and the image positions, respectively, normalized by the Einstein radius $\theta_E = \sqrt{4GM D_{LS}/c^2 D_L D_S}$.

§3. Perturbative approach

A comprehensive, algebraic study of the above lens equation Eqs. (2.6) and (2.7) was done by Asada⁹⁾ using polar coordinates. He showed the lens equation for this type as a single real 10th-order algebraic equation and gave analytic solutions

for a source located exactly on the principal axes. He also estimated the order of magnitude of the normalized quadrupole moment as

$$e \sim 10^{-5} \left(\frac{M_{\odot}}{M} \right) \left(\frac{R}{10^6 \text{ km}} \right)^3 \left(\frac{10^7 \text{ km}}{R_E} \right)^2 \left(\frac{v}{10 \text{ km s}^{-1}} \right)^2, \quad (3.1)$$

where R denotes the typical size of a lens object, R_E is the Einstein ring radius, and v is a rational surface velocity on the equatorial plane of the lens star. Then, for a nearby solar-type star at 10 pc, i.e., $R \sim 10^6$ km, $R_E \sim 10^7$ km, with a rotational velocity of $v \sim 10$ km s $^{-1}$ which is much faster than that of the Sun, the estimated value is $e \sim 10^{-5}$, which is sufficiently smaller than unity.

Because of the higher polynomial nature of the lens equations, however, general analytic solutions for a source not on the axes have not been given yet (or cannot be given for more than 4 images). Also, the physical properties of the lensed images, such as the image amplification factors, are not yet investigated.

Instead of such an exact algebraic treatment, in this paper, we take another, a more tractable approach to the quadrupole lens equations. Using the fact $0 < e \ll 1$, we employ a perturbative approach to the lens equations and obtain approximate solutions to the image positions (x, y) , up to the lowest order of the eigenvalue e of the quadrupole moment.

Concerning the perturbative approach to the gravitational lensing, for example, Alard^{(10)–(12)} wrote a series of papers in a slightly different context from ours. His main interest is the non-spherical perturbations of extended lens models and elongated arc images, whereas in this paper we concentrate our attention on a non-spherically distorted compact lens model and multiple images.

We start from the following set of the lens equation, which is obtained by clearing the fraction of Eqs. (2.6) and (2.7):

$$(x^2 + y^2)^2 \{ (x^2 + y^2)(\beta_x - x) + x \} = -e(x^2 - 3y^2)x, \quad (3.2)$$

$$(x^2 + y^2)^2 \{ (x^2 + y^2)(\beta_y - y) + y \} = -e(3x^2 - y^2)y. \quad (3.3)$$

Setting $e = 0$, we obtain the zeroth-order solutions,

$$x = x_0^{\pm} \equiv f^{\pm} \beta_x, \quad y = y_0^{\pm} \equiv f^{\pm} \beta_y, \quad (3.4)$$

where

$$f^{\pm} \equiv \frac{1 \pm \sqrt{1 + 4\beta^{-2}}}{2}, \quad (3.5)$$

and $\beta = \sqrt{\beta_x^2 + \beta_y^2}$. The above solutions are just the solutions for a point mass lens. We may also find a trivial solution $(x, y) = (0, 0)$, but it is inadequate because the denominators in the original lens equation Eqs. (2.6) and (2.7) vanish.

Next, we put the following form into Eqs. (3.2) and (3.3),

$$x = x_0 + x_1, \quad y = y_0 + y_1, \quad (3.6)$$

then, up to the linear order of x_1 and y_1 , we obtain

$$(2x_0x_1 + 2y_0y_1)(\beta_x - x_0) - (x_0^2 + y_0^2)x_1 + x_1 = -e \frac{(x_0^2 - 3y_0^2)x_0}{(x_0^2 + y_0^2)^2}, \quad (3.7)$$

$$(2x_0x_1 + 2y_0y_1)(\beta_y - y_0) - (x_0^2 + y_0^2)y_1 + y_1 = -e \frac{(3x_0^2 - y_0^2)y_0}{(x_0^2 + y_0^2)^2}. \quad (3.8)$$

The solutions are expressed in terms of β_x and β_y as follows:

$$x^\pm = x_0^\pm + x_1^\pm = f^\pm \beta_x + e \frac{(4\beta_x^2 - 3\beta_y^2)(f^\pm)^2 - 1}{\beta^2(f^\pm \beta^2 + 1)(f^\pm \beta^2 + 2)} \beta_x, \quad (3.9)$$

$$y^\pm = y_0^\pm + y_1^\pm = f^\pm \beta_y + e \frac{(3\beta_x^2 - 4\beta_y^2)(f^\pm)^2 + 1}{\beta^2(f^\pm \beta^2 + 1)(f^\pm \beta^2 + 2)} \beta_y. \quad (3.10)$$

In the case of a point mass lens model ($e = 0$), the number of the images is two. As shown by Asada,⁹⁾ the number of the images by a quadrupole lens ($e \neq 0$), which depends on the value of e and the position of the source, is generally more than two, in most cases four. We can obtain another, new solutions for $e > 0$ case, which are not existent for a point mass lens case, as a perturbation around the trivial solution $(x, y) = (0, 0)$, namely,

$$x = 0 + x_1, \quad y = 0 + y_1. \quad (3.11)$$

Up to the lowest non-trivial order, the lens equation is

$$(x_1^2 + y_1^2)^2 x_1 = -e(x_1^2 - 3y_1^2)x_1, \quad (3.12)$$

$$(x_1^2 + y_1^2)^2 y_1 = -e(3x_1^2 - y_1^2)y_1. \quad (3.13)$$

In the case of $x_1 \neq 0, y_1 \neq 0$, Eqs. (3.12) and (3.13) imply $x_1^2 = -y_1^2$, which does not have a real root. In the case $x_1 \neq 0, y_1 = 0$, Eq. (3.12) implies $x_1^2 = -e$, which again does not have a real root since we have assumed $e > 0$. Finally in the case $x_1 = 0, y_1 \neq 0$, Eq. (3.13) reduces to $y_1^2 = e$, from which we obtain

$$x = x_1 = 0, \quad y = y_1 = \pm\sqrt{e}. \quad (3.14)$$

The above solutions are always on the y -axis, independent of the source position (β_x, β_y) , and order of $O(\sqrt{e})$, not $O(e)$. Therefore, we perform one more iteration and obtain slightly higher order solutions in the following form,

$$x = 0 + x_2, \quad y = 0 + y_1 + y_2 = \pm\sqrt{e} + y_2. \quad (3.15)$$

Up to the lowest order of x_2 and y_2 , the lens equation is now

$$y_1^6 \beta_x + y_1^4 x_2 = 3ey_1^2 x_2, \quad (3.16)$$

$$y_1^6 (\beta_y - y_1) + 5y_1^4 y_2 = 3ey_1^2 y_2, \quad (3.17)$$

and the solution is

$$x = x_2 = \frac{1}{2}e\beta_x, \quad y = y_1 + y_2 = \pm\sqrt{e} \left(1 + \frac{1}{2}e\right) - \frac{1}{2}e\beta_y. \quad (3.18)$$

These images always appear very close to the y -axis, are very dim because the amplitude of the amplification factor is $O(e^2)$, and disappear when $e \rightarrow 0$. We call

Table I. Comparison of our approximate solutions with numerical ones for $\beta_x = 0$

e	β_x	β_y	x_{num}	x_{appr}	(1) y_{num}	(2) y_{appr}	$ (2) - (1) $
0.01	0.0	0.2	0.0	0.0	1.10087	1.10091	3.9×10^{-5}
			0.0	0.0	-0.89881	-0.89891	1.0×10^{-4}
			0.0	0.0	0.09950	0.099500	3.7×10^{-6}
			0.0	0.0	-0.10157	-0.10150	6.8×10^{-5}
0.01	0.0	0.5	0.0	0.0	1.27780	1.27782	1.8×10^{-5}
			0.0	0.0	-0.77262	-0.77282	2.0×10^{-4}
			0.0	0.0	0.09809	0.09800	8.5×10^{-5}
			0.0	0.0	-0.10327	-0.10300	2.7×10^{-4}
0.02	0.0	0.2	0.0	0.0	1.09668	1.09684	1.6×10^{-4}
			0.0	0.0	-0.89241	-0.89284	4.2×10^{-4}
			0.0	0.0	0.14084	0.14084	2.5×10^{-7}
			0.0	0.0	-0.14510	-0.14484	2.7×10^{-4}
0.02	0.0	0.5	0.0	0.0	1.27479	1.27486	7.3×10^{-5}
			0.0	0.0	-0.76402	-0.76486	8.4×10^{-4}
			0.0	0.0	0.13802	0.13784	1.8×10^{-4}
			0.0	0.0	-0.14878	-0.14784	9.5×10^{-4}

them as the “minor” images, and may safely neglect them in the analysis of the image amplifications.

In order to check the accuracy of our approximate solutions Eqs. (3·9), (3·10), and (3·18), we compare them with numerical solutions. Table I and II show the result for some sets of typical values of the parameters. Table I shows the calculated values for the source on the y -axis, i.e., $\beta_x = 0$, and Table II for the source on the x -axis, $\beta_y = 0$. The maximal error is about 10^{-4} , which is the same order of magnitude as $O(e^2)$.

Table II. Comparison of our approximate solutions with numerical ones for $\beta_y = 0$

e	β_x	β_y	(1) x_{num}	(2) x_{appr}	$ (2) - (1) $	(3) y_{num}	(4) y_{appr}	$ (4) - (3) $
0.01	0.2	0.0	1.10902	1.10906	3.8×10^{-5}	0.0	0.0	0.0
			-0.91097	-0.91106	9.7×10^{-5}	0.0	0.0	0.0
			0.00102	0.00100	2.0×10^{-5}	0.10048	0.10050	1.7×10^{-5}
			0.00102	0.00100	2.0×10^{-5}	-0.10048	-0.10050	1.7×10^{-5}
0.01	0.5	0.0	1.28372	1.28373	1.8×10^{-5}	0.0	0.0	0.0
			-0.78855	-0.78873	1.9×10^{-4}	0.0	0.0	0.0
			0.00254	0.00250	3.8×10^{-5}	0.10035	0.10050	1.5×10^{-4}
			0.00254	0.00250	3.8×10^{-5}	-0.10035	-0.10050	1.5×10^{-4}
0.02	0.2	0.0	1.11299	1.11314	1.5×10^{-4}	0.0	0.0	0.0
			-0.91676	-0.91714	3.8×10^{-4}	0.0	0.0	0.0
			0.00208	0.00200	8.1×10^{-5}	0.14281	0.14284	2.5×10^{-5}
			0.00208	0.00200	8.1×10^{-5}	-0.14281	-0.14284	2.5×10^{-5}
0.02	0.5	0.0	1.28662	1.28669	7.0×10^{-5}	0.0	0.0	0.0
			-0.79598	-0.79669	7.1×10^{-4}	0.0	0.0	0.0
			0.00516	0.00500	1.6×10^{-4}	0.14241	0.14284	4.2×10^{-4}
			0.00516	0.00500	1.6×10^{-4}	-0.14241	-0.14284	4.2×10^{-4}

§4. Amplification factor

Once we express the solutions for the image positions (x, y) in terms of the source position (β_x, β_y) , we can directly calculate the amplification factor from the Jacobian

$$A^\pm = \left| \det \frac{\partial(x^\pm, y^\pm)}{\partial(\beta_x, \beta_y)} \right|, \quad (4.1)$$

where A^+ and A^- represent the amplification factors for the images with positive and negative parity, respectively. The calculation is straightforward, and from Eqs. (3.9) and (3.10) we obtain the following results up to the linear order of e :

$$A^+ = A_0^+ \left(1 - e \frac{4(\beta_x^2 - \beta_y^2) (\beta(\beta^2 + 6) - (\beta^2 + 4)\sqrt{\beta^2 + 4})}{\beta^3(\beta^2 + 4) (\sqrt{\beta^2 + 4} + \beta)^2} \right), \quad (4.2)$$

$$A^- = A_0^- \left(1 - e \frac{4(\beta_x^2 - \beta_y^2) (\beta(\beta^2 + 6) + (\beta^2 + 4)\sqrt{\beta^2 + 4})}{\beta^3(\beta^2 + 4) (\sqrt{\beta^2 + 4} - \beta)^2} \right), \quad (4.3)$$

where

$$A_0^\pm \equiv \frac{(\sqrt{\beta^2 + 4} \pm \beta)^2}{4\beta\sqrt{\beta^2 + 4}} \quad (4.4)$$

represent the amplification factors for a point mass lens, namely $e = 0$ case.

We can also define the Padé approximants for A^\pm of order $[0/1]^*$ with respect to e as follows:

$$A_P^+ \equiv A_0^+ \left(1 + e \frac{4(\beta_x^2 - \beta_y^2) (\beta(\beta^2 + 6) - (\beta^2 + 4)\sqrt{\beta^2 + 4})}{\beta^3(\beta^2 + 4) (\sqrt{\beta^2 + 4} + \beta)^2} \right)^{-1}, \quad (4.5)$$

$$A_P^- \equiv A_0^- \left(1 + e \frac{4(\beta_x^2 - \beta_y^2) (\beta(\beta^2 + 6) + (\beta^2 + 4)\sqrt{\beta^2 + 4})}{\beta^3(\beta^2 + 4) (\sqrt{\beta^2 + 4} - \beta)^2} \right)^{-1}. \quad (4.6)$$

As shown in Table III, the Padé approximants Eqs. (4.5) and (4.6) generally give the approximate amplifications with better accuracy of about 1% or less. Hereafter, we use Eqs. (4.5) and (4.6) as our approximate formulae for the amplification factors.

*) Generally speaking, the Padé approximant of order $[m/n]$ is an approximation of a function by a ratio of two polynomials of order m and n . Then, for a function $A(x)$ whose Taylor expansion is given by $A(x) \simeq A_0(1 + a_1x + \dots)$, the Padé approximant of order $[0/1]$ is simply $A_0/(1 - a_1x)$. At the same order, Padé approximations are usually superior to Taylor expansions especially when functions contain poles.

§5. Changes in the image properties

For a point mass lens model, we have the following “universal” relations for the image properties. First, the sum of the two image positions is

$$x_0^+ + x_0^- = \beta_x, \quad y_0^+ + y_0^- = \beta_y. \quad (5.1)$$

Namely, the sum of the image positions is always equal to the original source position. Second, the difference of the amplification factors is always unity:

$$A_0^{\text{diff}} \equiv A_0^+ - A_0^- = 1. \quad (5.2)$$

This also means, for a point mass lens, the image with positive parity is always brighter than that with negative parity. Just for reference, we also show that the image separation and the total amplification are expressed in the following way:

$$\Delta x_0 \equiv x_0^+ - x_0^- = \frac{\sqrt{\beta^2 + 4}}{\beta} \beta_x, \quad \Delta y_0 \equiv y_0^+ - y_0^- = \frac{\sqrt{\beta^2 + 4}}{\beta} \beta_y, \quad (5.3)$$

$$A_0^{\text{tot}} \equiv A_0^+ + A_0^- = \frac{\beta^2 + 2}{\beta \sqrt{\beta^2 + 4}}. \quad (5.4)$$

In this section, we investigate how the quadrupole moment changes the above image properties. From Eqs. (3.9) and (3.10), we obtain the sum as

$$x^+ + x^- = \beta_x - e \left(1 + \frac{4\beta_y^2}{\beta^4} \right) \beta_x, \quad y^+ + y^- = \beta_y + e \left(1 + \frac{4\beta_x^2}{\beta^4} \right) \beta_y, \quad (5.5)$$

The change due to e is getting larger as $\propto 4e\beta^{-2}$ for $|\beta| \ll 1$. Therefore, even if $|e| \ll 1$, the “universal” relation Eq. (5.1) may be significantly broken if $|\beta| \ll 1$.

Table III. Comparison of our approximate amplification factors with numerical ones

e	β_x	β_y	parity	(1) A_{num}	(2) A	$ (2) - (1) /(1)$	(3) A_P	$ (3) - (1) /(1)$
0.01	0.0	0.2	+	2.83741	2.82454	0.45%	2.83847	0.04%
			−	2.37779	2.32454	2.24%	2.37166	0.26%
0.01	0.0	0.5	+	1.56603	1.56568	0.02%	1.56609	0.00%
			−	0.65303	0.64568	1.13%	0.65116	0.29%
0.02	0.0	0.2	+	2.66004	2.61173	1.82%	2.66404	0.15%
			−	2.87313	2.61173	9.10%	2.83724	1.25%
0.02	0.0	0.5	+	1.54134	1.53994	0.09%	1.54156	0.01%
			−	0.73395	0.69994	4.63%	0.72434	1.31%
0.01	0.2	0.0	+	3.26482	3.25015	0.45%	3.26619	0.04%
			−	1.78910	1.75015	2.18%	1.78564	0.19%
0.01	0.5	0.0	+	1.61749	1.61714	0.02%	1.61758	0.00%
			−	0.54292	0.53714	1.06%	0.54170	0.22%
0.02	0.2	0.0	+	3.52534	3.46296	1.77%	3.53232	0.20%
			−	1.59981	1.46296	8.55%	1.58928	0.66%
0.02	0.5	0.0	+	1.64434	1.64288	0.09%	1.64460	0.02%
			−	0.50371	0.48288	4.14%	0.49971	0.80%

On the contrary, the image separation is

$$\Delta x \equiv x^+ - x^- = \Delta x_0 \left\{ 1 - \left(\frac{2(\beta_x^2 - \beta_y^2)}{\beta^2(\beta^2 + 4)} - 1 \right) e \right\}, \quad (5.6)$$

$$\Delta y \equiv y^+ - y^- = \Delta y_0 \left\{ 1 - \left(\frac{2(\beta_x^2 - \beta_y^2)}{\beta^2(\beta^2 + 4)} + 1 \right) e \right\}. \quad (5.7)$$

It is evident that, even if $|\beta| \ll 1$, the change due to e in the image separation is $\sim e \times O(1)$. Therefore, we may conclude that the change due to e in the image separation is small for $|e| \ll 1$.

Similar situation occurs in the analysis of the image amplification. From Eqs. (4.5) and (4.6), the amplification difference for $|\beta| \ll 1$ is

$$A^{\text{diff}} \equiv A_P^+ - A_P^- \simeq \frac{1 + \frac{2}{\beta^2} e'}{1 + \frac{7}{2} e' - \left(\frac{2}{\beta} e' \right)^2}, \quad (5.8)$$

where

$$e' = \left(\left(\frac{\beta_x}{\beta} \right)^2 - \left(\frac{\beta_y}{\beta} \right)^2 \right) e. \quad (5.9)$$

The change due to e is getting larger as $\propto 2e\beta^{-2}$ for $|\beta| \ll 1$. Therefore, even if $|e| \ll 1$, the “universal” relation Eq. (5.2) may be significantly broken if $|\beta| \ll 1$. As is already shown numerically in Table III, there is even a case of $A^{\text{diff}} < 0$ for $e = 0.02, \beta_x = 0.0, \beta_y = 0.2$.

The total amplification is

$$A^{\text{tot}} \equiv A_P^+ + A_P^- \simeq \frac{A_0^{\text{tot}}}{1 + \frac{3}{2} e' - \left(\frac{2}{\beta} e' \right)^2}. \quad (5.10)$$

Compared to A^{diff} , the change due to e is relatively mild for $|\beta| \ll 1$.

We give some comments on the range of the parameters e and β . Since we have started from the linear perturbation with respect to e , the range of validity is $0 \leq e \ll 1$. If we assume additionally that the maximal error in the amplification factors should be less than, say, 1%, the range of validity may be $0 \leq e \lesssim 0.01$, which we can guess from Table III. The ranges of β_x and β_y are more important. From Eqs. (5.8) and (5.10), we can obtain the constraint $|e'| \lesssim \beta^2$ and $|e'| \ll \beta$. If not, then the term of $O(e^2)$ in the denominators of Eqs. (5.8) and (5.10) can make a dominant contribution over the linear term, which is outside of the validity of linear perturbation theory.

§6. Summary

In this paper, we have investigated the gravitational lens effect caused by a non-spherical, compact object. The non-spherical property of the gravitational potential

is expressed by the quadrupole moment. Eqs. (2.6) and (2.7) are the lens equations for a compact lens object with a quadrupole moment. In these lens equations, the effect of the non-spherical contribution is expressed by e , the eigenvalue of the normalized quadrupole moment tensor \tilde{Q}_{ij} .

Under the assumption that the quadrupole moment e is small, we have perturbatively solved the lens equation and obtained the solutions for the image positions. Eqs. (3.9) and (3.10) represent the perturbative solutions for the positions of the “major” images, which are reduced to the solutions for a point mass lens model in the limit $e \rightarrow 0$. We have also found two new perturbative solutions for the “minor” images, which are given in Eq. (3.18). The minor images are very dim, always appear very close to the y -axis, near $(0, \pm\sqrt{e})$, and vanish in the limit $e \rightarrow 0$. The accuracy of our approximate solutions is numerically checked and summarized in Tables I and II for some sets of parameters. It is shown that the maximal error in the image position is about 10^{-4} , which is the same order of magnitude as $O(e^2)$.

Then, we have calculated the amplification factors for the major images from the Jacobian and obtained the Padé approximants of them in Eqs. (4.5) and (4.6). The accuracy of our approximate formula for the amplification factor is numerically estimated and summarized in Table III for some sets of parameters. It is shown that the typical relative error is 1% or less.

For a point mass lens model, we know the simple “universal” relations for the image properties, Eqs. (5.1) and (5.2). We have investigated how the quadrupole contribution e changes such “universal” relations. We have found that the change due to a non-zero e in the image separation is $\sim e \times O(1)$. However, the difference of the amplifications $A^{\text{diff}} = A^+ - A^-$, which is always unity in the case of a point mass lens, may be significantly changed due to the quadrupole contribution, which is shown in Eq. (5.8).

Mao and Schneider¹³⁾ discussed the “anomalous” flux ratio between the images caused by a lens galaxy. The lens system they discussed was QSO B 1422 + 231. Whereas the well-known “simple” lens models, such as a singular isothermal ellipsoid, could fit the observed image positions very accurately, they all failed to obtain the observed flux ratios. The possibility they argued is that the discrepancy between the observed and model-predicted flux ratios is due to substructure or perturbation in the simple lens models, which only slightly changes the image positions but significantly affects the flux ratios. Since the lens model in this paper is different from that of Mao and Schneider,¹³⁾ we have no intention of directly applying our results to their lens systems. Still, it should be noted that the results in our paper show some formal similarity. It indicates that even a tiny non-spherical distortion of the lens potential may also cause significant amount of flux anomalies in the lensed images, whereas it only slightly changes the image positions.

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